On the existence of almost Fano 3-folds with del Pezzo fibrations

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On the existence of almost Fano 3-folds with del Pezzo fibrations (arXiv:1603.07139)

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Motivation and Main Result

Our motivation is **the classification of smooth almost Fano 3-folds X with ray contraction $\varphi\colon X \to \mathbb{P}^1$: del Pezzo fibration of deg d (=dP_d-fib)**.

- X is almost Fano $\iff$ $-K_X$ is nef and big but not ample.
- $\varphi\colon X \to \mathbb{P}^1$ is dP_d-fib $\iff$ gen'l φ-fib is del Pezzo surf. of deg d.

To classify such 3-folds, we are interested in the following questions:

Questions

Q1 Narrow down the possible value of invariants of such 3-folds. As invariants, we consider $(-K_X)^3$, $h^{1,2}(X)$ and types of contractions of extremal rays.
Q2 For each possible values, find a such 3-fold having it as whose invariants.

[JPR05,11] and [Tak09] classified the possibilities of almost Fano 3-folds with dP_d-fib. However $\exists$ **exactly 10 cases s.t. the existence of an example is unknown.**

Main Theorem [Fuk16]

For each of 10 cases (=Table 1,2), there exists an example.

Setting

X : almost Fano 3-fold with dP_d-fib $\varphi\colon X \to \mathbb{P}^1$. $\psi\colon X \to \overline{X}$: contraction of K_X-triv. ray. Then ψ is a div'l contraction (Case (A)) or flopping (Case (B)).
In Case (B), X^+ : the flop of X. $\varphi^+\colon X^+ \to V^+$: the contraction of K_{X^+}-neg. ray.

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi : \text{div'l}"] & \\ \mathbb{P}^1 & & \overline{X} \end{tikzcd}$$

Figure 1: Case (A)

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[rr, dashed, "\chi : \text{flop}"] & & X^+ \arrow[dl, "\psi^+"'] \arrow[dr, "\varphi^+"] & \\ \mathbb{P}^1 & & \overline{X} & & V^+ \end{tikzcd}$$

Figure 2: Case (B)

Table 1: **Case (A)**

Name	$\varphi : dP_d$	Type of $\psi\colon X \to \overline{X}$	$(-K_X)^3$	$h^{1,2}(X)$
(A-1)	6	$(g,d)=(1,6)$	12	$\exists 2$
(A-2)	5	$(g,d)=(1,5)$	10	$\exists 6$

Table 2: **Case (B)**

Name	$\varphi : dP_d$	V^+	$\varphi^+\colon X^+ \to V^+$	$(-K_X)^3$	$h^{1,2}(X)$
(B-i-1)	6	$B(4)$	$(g,d)=(1,6)$	8	3
(B-i-2)	6	$V(10)$	$(g,d)=(1,6)$	6	3
(B-i-3)	6	$V(9)$	$(g,d)=(1,6)$	4	4
(B-ii)	6	$\mathbb{P}^2$	$\deg(\text{disc.})=4$	14	$\exists 2$
(B-iii-1)	6	$\mathbb{P}^1$	dP_6	12	$\exists 2$
(B-iii-2)	6	$\mathbb{P}^1$	dP_6	6	$\exists 4$
(B-iii-3)	6	$\mathbb{P}^1$	dP_6	4	$\exists 3$
(B-iii-4)	6	$\mathbb{P}^1$	dP_6	2	$\exists 5$

Notation for TABLE 1 : Case (A)

- In the 3rd row, $\underline{(g,d)}$ means that ψ is the blowing-up along a curve C of genus g with $(-K_{\overline{X}}).C = d$.
- In the last row, $\underline{\exists n}$ means that $\exists$ at least one member X with $h^{1,2}(X)=n$.

Notation for TABLE 2 : Case (B)

- The 3rd row denotes the types of V^+. $B(m)$ (resp. $V(g)$) denotes a del Pezzo (resp. Mukai) threefold of degree m (resp. genus g).
- The 4th row denotes the types of φ^+.
  - $\underline{(g,d)}$ means that φ^+ is the blup along a curve C of genus g with $-K_{V^+}.C = i_{V^+}d$. Here i_{V^+} denotes the Fano index of V^+.
  - $\underline{\text{deg(disc.)}=4}$ means that φ^+ is a conic bundle with a deg 4 discriminant.
  - $\underline{dP_6}$ means that φ^+ is a dP_6-fib.
- In the last row, $\underline{\exists n}$ means that $\exists$ at least one member X with $h^{1,2}(X)=n$.

Known Results

- Jahnke-Peternell-Radloff treated Case (A) and Case (B) [JPR05,11]. They narrow down possible values of inv.s and show that $\exists$ such 3-folds except for some cases.
- Takeuchi treated Case (B) with $d \neq 6$ [Tak09]. He classified the possible values of invariants and show that $\exists$ such 3-folds.

These works and Main thm answer Q1 and Q2 for almost Fano 3-folds w/ dP_d-fib.

Outline of proof

- <u>Cases (B-i)</u> The problems are reduced to the $\exists$ of an elliptic curve of deg 6 in $\underline{V^+ = B(4)}, V(10), V(9)$. In the case (B-i-1), we can construct such a curve in $B(4) = (2)^2 \subset \mathbb{P}^5$ directly. In other cases, we can prove it by using "surjectivity of a period map" for NS lattice for K3 (cf [Mor84]) and Mukai's theory [Muk95]. This method was also employed in [CM13].
- <u>Cases (A-1),(B-ii),(B-iii)</u>

Idea

A quadric surface Q with 3 general pts has a conic through the 3 pts. By blowing up the 3 pts and contracting the conic, we get a dP_6.

This transform can be relativized as the following lemma:

Lemma

Let $\pi\colon W \to \mathbb{P}^1$ be a quadric fibration and $B \subset W$ a smooth curve. Assume that $(\pi\colon W \to \mathbb{P}^1, B)$ satisfy (♣) :

(♣) $\deg(\pi|_B) = 3$ and $-K_Z$ is nef big / $\mathbb{P}^1$ where $Z := \mathrm{Bl}_B W$.

Then $\exists$ the following diagram with the following properties.

$$\begin{tikzcd} Y \arrow[d, "\mu"'] & & Z \arrow[ll, dashed, "\Phi"'] \arrow[d, "\tau"] \\ X \arrow[r, "\varphi"'] & \mathbb{P}^1 & W \arrow[l, "\pi"] \end{tikzcd}$$

- Φ is isom. in codim.1 / $\mathbb{P}^1$ (More precisely, id or flop / $\mathbb{P}^1$).
- φ is a dP_6-fib and μ is a blowing-up along a φ-section C.

By using this lemma, we can construct a pair $(\pi\colon W \to \mathbb{P}^1, B)$ satisfying (♣) with the output X is the example.
• <u>Case (A-2)</u> We can prepare a similar lemma and construct an example of (A-2) by using it.

Relative linear extension of dP_6-fib

$\varphi\colon X \to C$: dP_d-fib. If $d \neq 6$ then $\exists$ isotrivial fibration of Fano variety $\varphi_Y\colon Y \to C$ containing X as a C.I. / C. For example, if $d=3$ then we can take $\varphi_Y = \mathbb{P}^3$-bdl.

Question

Assume $d=6$. Does $\exists \varphi_Y\colon Y \to C$: $(\mathbb{P}^2)^2$ or $(\mathbb{P}^1)^3$-fibration containing X as a linear section / C ?

I think this question is a factor of the difficulty of construction of examples for dP_6-fib. Note that if $C = \mathbb{P}^1$ then such φ_Y <u>MUST</u> have singular fibers. We can prove that this question holds for some almost Fano 3-folds with dP_6-fib.

Example

For $\forall$ almost Fano 3-fold X with dP_6-fib with $(-K_X)^3 = 22, 16, 8$ (resp. 18), $\exists (\mathbb{P}^2)^2$ (resp. $(\mathbb{P}^1)^3$)-fib. $\varphi_Y\colon Y \to \mathbb{P}^1$ containing X as a linear section / $\mathbb{P}^1$. Moreover, if $(-K_X)^3 = 22$ then φ_Y is not depend on isom. class of X.

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